After The Atlas: Convexity, Continuity, And Law-Invariance

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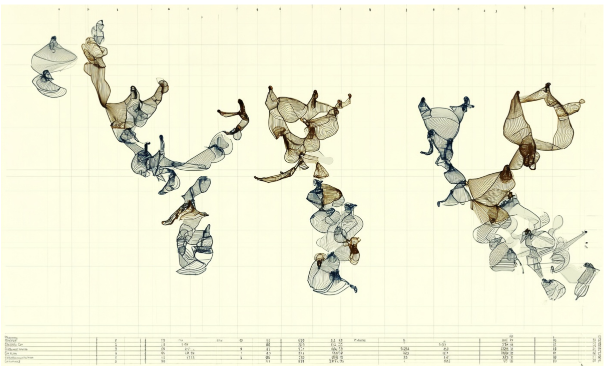
Abstract

This note extends An Atlas of Monetary Risk Measures. The Atlas organizes monetary risk measures by algebraic behavior and continuity. Two refinements create a more useful map. First, Frittelli and Rosazza Gianin (2002) show that convexity plus lower semicontinuity is the structural dimension: every such functional is the supremum of affine functionals with a penalty, and the familiar risk-measure axioms appear as restrictions on the dual variables and penalties. Second, Liu et al. (2020) show that law-invariant convex risk functionals admit a parallel representation in which arbitrary statewise dual variables are replaced by rank-based signed Choquet support functions indexed by concave functions h. The resulting class includes monetary risk measures, deviation measures, mean-variance principles, Gini-type functionals, and other non-monotone objects that sit outside the monetary Atlas.



Introduction

This note extends the Atlas. The Atlas uses a two-dimensional geometry: the horizontal direction tracks algebraic structure, from additive to comonotonic additive, coherent, convex, and monetary, while the vertical direction tracks continuity and law invariance. Two points deserve more emphasis.

The first point is due to Frittelli and Rosazza Gianin (2002). Their paper identifies the clean separation between the convex analytic core and the risk-measure axioms. The analytic core is convexity plus lower semicontinuity. Once those two assumptions hold, Fenchel–Moreau duality gives a representation as a supremum of penalized linear functionals. Other risk measure axioms, such as monotonicity, cash invariance, normalization, positive homogeneity, and coherence then become restrictions on the dual variables and on the penalty. The Atlas uses that structure but fails to emphasize its importance as a organizing principle.

The second point is developed in Liu et al. (2020). Their law-invariant convex risk functionals enlarge the picture beyond monetary risk measures. They keep translation behavior, convexity, continuity, and law invariance, but they do not require monotonicity, and the translation coefficient need not be one. The result is a much larger family that includes risk measures, deviation measures, mean-variance principles, standard-deviation principles, Gini-type functionals, signed Choquet integrals, and other variability functionals. Many of these objects look like ordinary distortion risk measures after two small changes: the distortion no longer has to map $[0, 1]$ into $[0, 1]$, and the support function may be signed rather than monotone.

The original Atlas intentionally focuses on monetary risk measures, because monotonicity and cash invariance are the minimal conditions for interpreting $\rho(X)$ as a premium or consideration for accepting a loss X. Liu et al. (2020) show what lies outside that monetary world. Their framework is especially useful because it uses the same convex-dual machine, but with rank-based support functions rather than arbitrary statewise dual scenarios.

Notation And Conventions

We keep the Atlas conventions. We work on an atomless probability space when law invariance matters. Random variables are losses. Larger X is worse. We write $\rho(X)$ for a functional on a suitable L^p space, and ρ is monetary when it is monotone and cash invariant:

$$X \le Y \implies \rho(X) \le \rho(Y), \qquad \rho(X+m) = \rho(X) + m.$$

We write $q_X(u)$ for the increasing quantile of X, $0 < u < 1$. Liu et al. use the upper-tail convention

$$\mathsf{VaR}_\alpha(X) = \inf\{\, x : \mathsf{P}\{X > x\} \le \alpha \,\}, \qquad 0 \le \alpha \le 1.$$

Thus, in Atlas notation, away from atoms and endpoint conventions,

$$\mathsf{VaR}_\alpha(X) = q_X(1-\alpha).$$

It is convenient to write

$$r_X(\alpha) := q_X(1-\alpha),$$

so that r_X is the decreasing tail-quantile function used by Liu et al. A concave function $h : [0,1] \to \mathbb{R}$ with $h(0) = 0$ defines the signed Choquet support functional

$$\rho_h(X) := \int_0^1 r_X(\alpha)\, dh(\alpha).$$

When h is absolutely continuous, the same functional is

$$\rho_h(X) = \int_0^1 r_X(\alpha) h'(\alpha) d\alpha = \int_0^1 q_X(u) h'(1-u) du.$$

The last display is the Atlas version: a spectral-looking integral against q_X, but with weight $u \mapsto h'(1-u)$ that need not be nonnegative and need not integrate to one unless we impose those restrictions.

For $c \in \mathbb{R}$, let

$$\Phi_c := \{\, h : h \text{ is concave on } [0,1],\ h(0) = 0,\ h(1) = c \,\}.$$

For L^p, Liu et al. impose the corresponding integrability condition on h', writing Φ_c^p for the admissible class. On L^∞, every concave h in Φ_c is admissible.

The number $c = h(1)$ is is the translation coefficient:

$$\rho_h(X+m) = \rho_h(X) + cm.$$

Classical monetary risk measures have $c = 1$. Deviation measures have $c = 0$. Mean-variance and standard-deviation principles have $c = 1$ if they include the mean term, and their pure deviation components have $c = 0$.

The Frittelli Point: Convex Plus Continuity Is The Structural Dimension

Frittelli and Rosazza Gianin start with a functional p on an ordered locally convex space $\mathcal{X}$. In Atlas loss notation, we may think of p and ρ as the same object up to the sign convention used for final wealth. Their Assumption A is finite-valued convexity plus lower semicontinuity. Under that assumption, the functional has the representation

$$p(x) = \sup_{x' \in X'} \{\, x'(x) - F(x') \,\},$$

for a convex penalty F on the continuous dual space. Conversely, every such representation gives a convex lower-semicontinuous functional. If the penalty is zero on its effective domain and infinite outside it, the representation reduces to a support function, and the represented functional is sublinear.

The important point is that the representation does not come from monotonicity, cash invariance, or coherence. It comes from convexity and lower semicontinuity alone. The remaining axioms tell us which dual variables are allowed and how the penalty behaves. In the monetary loss convention, the familiar representation reads

$$\rho(X) = \sup_{Z \in \mathcal{D}} \{ \mathsf{P}(XZ) - \alpha(Z) \} .$$

The role of each axiom is then visible directly in the dual ingredients.

Assumption On ρ	Dual Consequence
Convex and lower semicontinuous	ρ is the supremum of penalized continuous linear functionals.
Positive homogeneity	α is an indicator: 0 on a convex set, ∞ outside it.
Monotonicity	Effective dual variables are positive: $Z \geq 0$.
Cash invariance with coefficient c	Effective dual variables have mass $\mathsf{P}Z = c$.
Monetary normalization	$c = 1$ and $\inf_Z \alpha(Z) = 0$.
Coherence	$Z \geq 0$, $\mathsf{P}Z = 1$, and α is an indicator.
Fatou property on L^∞	The dual variables may be taken in L^1 rather than in ba.
Law invariance	The dual description is invariant under rearrangements, hence becomes quantile- or orbit-based.

This table is the two-dimensional Atlas in dual form. The first row is the structural axis: convexity plus continuity puts us in convex duality. The other rows are not separate representation theorems; they are restrictions on the support function or penalty.

This view also clarifies why coherent risk measures are not a natural starting point. Coherence is the special case in which the penalty is flat. Convex risk measures are not coherent risk measures with a defect; coherent risk measures are convex risk measures with a degenerate penalty. In pricing language, a coherent risk measure charges the worst expected loss over a set of scenarios. A convex risk measure charges the worst corrected expected loss, where the correction penalizes less plausible or less relevant scenarios.

The Liu Point: Law Invariance Turns Densities Into Concave Rank Functions

For law-invariant functionals, the dual variables cannot matter through their names as states. Only their distributions, or equivalently their quantile functions, can matter. Liu et al. push that observation to its natural conclusion.

Start from Fenchel–Moreau on L^p:

$$\rho(X) = \sup_{Z \in L^q} \{ \mathsf{P}(XZ) - \rho^*(Z) \} .$$

Translation behavior with coefficient c forces $\mathsf{P}Z = c$ on the effective domain. Law invariance allows us to rearrange X against Z and keep only the best coupling. The rearrangement step is the Hardy–Littlewood inequality, also called the Frechet–Hoeffding inequality in this context:

$$\sup_{Y \sim X} \mathsf{P}(YZ) = \int_0^1 r_X(\alpha) r_Z(\alpha) d\alpha.$$

For such a Z, define

$$h_Z(\alpha) := \int_0^\alpha r_Z(t) dt.$$

Because r_Z is decreasing, h_Z is concave, $h_Z(0) = 0$, and $h_Z(1) = \mathsf{P}Z = c$. The dual variable Z is therefore replaced by the concave function h_Z. The support functional becomes

$$\int_0^1 r_X(\alpha)dh_Z(\alpha).$$

The Liu representation is therefore

$$\rho(X) = \sup_{h\in\Phi_c^p} \left\{ \int_0^1 r_X(\alpha)dh(\alpha) - \beta(h) \right\}.$$

The penalty β is simply the conjugate penalty ρ^* pushed through the map $Z \mapsto h_Z$. Positive homogeneity again collapses β to an indicator, giving

$$\rho(X) = \sup_{h\in\Psi_c^p} \int_0^1 r_X(\alpha)dh(\alpha).$$

That formula is the law-invariant version of the Frittelli representation. General convex lower-semicontinuous functionals are supported by statewise linear functionals Z. Law-invariant convex functionals are supported by ranked linear functionals h.

Distortions Without The Probability Straightjacket

Classical distortion risk measures require a distortion $g : [0,1] \to [0,1]$ with $g(0) = 0$ and $g(1) = 1$, usually increasing and concave. The range condition has two jobs. First, $g(1) = 1$ gives cash invariance with coefficient one. Second, monotonicity of g gives monotonicity of the resulting risk measure.

Liu et al. remove both restrictions. Their h belongs to Φ_c for any $c \in \mathbb{R}$. It need not be increasing, and its values need not lie in $[0,1]$. Concavity is retained because it is the rank-space shadow of convexity: a decreasing rearranged dual density integrates to a concave primitive. The resulting signed Choquet integral still behaves like a distortion functional, but its support weight may be signed and its total mass may be $c \neq 1$.

The classical cases reappear by imposing restrictions on h.

Restriction On h	Resulting Functional Class
$h \in \Phi_1$, h increasing	Monotone law-invariant risk support functional.
$h \in \Phi_1$, h increasing and concave	Concave distortion or spectral support functional.
$h \in \Phi_0$	Pure deviation or variability support functional.
$h \in \Phi_c$ with $c \neq 0, 1$	Translation coefficient c, not a monetary risk measure unless $c = 1$.
h not increasing	Signed Choquet support, generally non-monotone.
Supremum over h with penalty β	Convex law-invariant functional.
Supremum over h with no finite penalty	Positively homogeneous law-invariant functional.

This enlarged distortion view is helpful because it treats risk measures and deviation measures with the same notation. The mean is the support functional for $h(\alpha) = \alpha$. TVaR is the support functional for the kinked concave function $h(\alpha) = \min\{\alpha/(1-p), 1\}$, where p is the Atlas lower-tail confidence level. Standard deviation is the supremum of signed Choquet supports with $h \in \Phi_0$ and an L^2 bound on h'. Variance is the same family with a quadratic penalty instead of an indicator.

Support Functions In The Liu Representation

The following table records common functionals in Liu form. We write

$$S_h(X) := \int_0^1 r_X(\alpha)dh(\alpha), \qquad r_X(\alpha) = q_X(1-\alpha).$$

When h is absolutely continuous, $S_h(X) = \int_0^1 q_X(u)h'(1-u)du$. We write q for the Holder conjugate of p when an L^p deviation is involved.

Table 3: Common Functionals Written In Liu Support Form

Functional	Formula	Liu Support Form	Translation Coefficient	Monotone?
Generic law-invariant convex risk functional	$\rho(X)$	$\sup_{h\in\Phi_c^p}\{\, S_h(X)-\beta(h)\,\}$	c	Not necessarily
Positive homogeneous law-invariant convex functional	$\rho(\lambda X)=\lambda\rho(X)$	$\sup_{h\in\Psi_c^p} S_h(X)$	c	Not necessarily
Mean	$\mathsf{P}X$	$S_h(X)$ with $h(\alpha)=\alpha$	1	Yes
Signed Choquet support	$\int X d(h\circ\mathsf{P})$	$S_h(X)$ for fixed $h\in\Phi_c$	c	iff h is increasing
Concave distortion risk measure	$\rho_g(X)=\int X d(g\circ\mathsf{P})$	$S_g(X)$ with $g\in\Phi_1$, g increasing concave	1	Yes
Spectral risk measure	$\int_0^1 q_X(u)\phi(u)du$	$S_h(X)$ with $h'(1-u)=\phi(u)$, $\phi\geq 0$ increasing, $\int_0^1\phi=1$	1	Yes
TVaR at Atlas level p	$(1-p)^{-1}\int_p^1 q_X(u)du$	$S_h(X)$ with $h(\alpha)=\min\{\alpha/(1-p),1\}$	1	Yes
Ess sup	$\operatorname{ess\,sup} X$	Endpoint limit of TVaR, equivalently mass at $\alpha=0$	1	Yes
Law-invariant coherent risk measure	Supremum of spectral supports	$\sup_{h\in\Psi_1} S_h(X)$ with h increasing concave	1	Yes
Law-invariant convex risk measure	Penalized supremum of spectral supports	$\sup_{h\in\Phi_1}\{\, S_h(X)-\beta(h)\,\}$ with h increasing	1	Yes
Standard deviation	$\sigma(X)$	$\sup\{\, S_h(X): h\in\Phi_0^2,\ \lVert h'\rVert_2\leq 1\,\}$	0	No
Variance	$\sigma^2(X)$	$\sup_{h\in\Phi_0^2}\{\, S_h(X)-\frac{1}{4}\lVert h'\rVert_2^2\,\}$	0	No
Mean-variance principle	$c\mathsf{P}X+a\sigma^2(X)$, $a>0$	$\sup_{h\in\Phi_c^2}\{\, S_h(X)-\frac{1}{4a}\lVert h'-c\rVert_2^2\,\}$	c	Generally no
Standard-deviation principle	$\mathsf{P}X+a\sigma(X)$	$\sup\{\, S_h(X): h\in\Phi_1^2,\ \lVert h'-1\rVert_2\leq a\,\}$	1	Only under additional bounds
Mean-L^p deviation	$\mathsf{P}X+a\lVert X-\mathsf{P}X\rVert_p$	$\sup\{\, S_h(X): h\in\Phi_1^p,\ \lVert h'-1\rVert_q\leq a\,\}$	1	Generally no
Mean absolute deviation	$\mathsf{P}X+a\mathsf{P}\lvert X-\mathsf{P}X\rvert$	The mean-L^p row with $p=1$, $q=\infty$	1	Generally no
Mean p-semi-deviation	$\mathsf{P}X+a\lVert(X-\mathsf{P}X)^+\rVert_p$	S_h generated by $Z=1+W-\mathsf{P}W$, $W\geq 0$, $\lVert W\rVert_q\leq a$	1	Yes for the usual coefficient bounds
Gini mean difference	$\mathsf{P}\lvert X_1-X_2\rvert$	$S_h(X)$ with $h(\alpha)=2\alpha(1-\alpha)$, up to the chosen Gini normalization	0	No

Functional	Formula	Liu Support Form	Translation Coefficient	Monotone?
Mean-Gini principle	$\mathsf{P}X + a\mathsf{P}\lvert X_1 - X_2\rvert$	S_h generated by $Z \preceq_{cx} 1 + a(2U-1)$, $U \sim U(0,1)$	1	Yes for the usual coefficient bounds
Range	$\operatorname{ess\,sup} X - \operatorname{ess\,inf} X$	Signed endpoint support with positive mass at $\alpha = 0$ and negative mass at $\alpha = 1$	0	No
Gini shortfall	Gini-type tail shortfall	Single signed Choquet support S_h, with h from the Gini-shortfall distortion	Depends on $h(1)$	Not necessarily
Wang premium principle	$\int X d(g \circ \mathsf{P})$	$S_g(X)$ for Wang distortion g	$g(1)$	Yes when g is increasing
Entropic risk measure	$\theta^{-1} \log \mathsf{P}e^{\theta X}$	In Frittelli form, $\sup_{Z \geq 0, \mathsf{P}Z=1} \{ \mathsf{P}(XZ) - \theta^{-1}\mathsf{P}(Z \log Z) \}$; in Liu form, push Z to h_Z and push entropy to $\beta(h_Z)$	1	Yes
Optimized certainty equivalent	$\inf_m \{ m + \mathsf{P}\ell(X - m) \}$	Frittelli penalty $\mathsf{P}\ell^*(Z)$; Liu penalty after $Z \mapsto h_Z$	1 under the standard normalization	iff ℓ is nondecreasing

The table mixes two kinds of entries. Some rows are single supports S_h: they are comonotonic additive. Other rows are suprema of supports, either with an indicator penalty or with a genuine convex penalty. The difference mirrors the Atlas distinction between comonotonic additive, coherent, and convex.

The variance row is the most revealing example. Standard deviation is positively homogeneous, so its Liu representation is a support function over an L^2 ball of signed rank weights. Variance is convex but not positively homogeneous, so its Liu representation uses the same signed rank weights but charges a quadratic penalty. Mean-variance adds a cash coefficient by shifting the derivative from h' to $h' - c$.

Relation To The Kusuoka Corner Of The Atlas

Kusuoka's theorem says that law-invariant convex monetary risk measures with the appropriate continuity are built from TVaR, or equivalently from penalized spectral risk measures. Liu et al. extend the same logic beyond monotone monetary functionals. In the monetary case, the support functions are increasing concave distortions with $h(1) = 1$. In the wider convex-risk-functional case, the support functions are all concave h with $h(1) = c$, including signed supports.

Thus the Liu theorem extends Kusuoka to a larger rank-based representation in which Kusuoka is the monotone, cash-invariant sub-case. The Atlas's bottom row can therefore be read as follows:

Region	Support Functions
Spectral/comonotonic additive	One increasing concave $h \in \Phi_1$.
Law-invariant coherent	Supremum over increasing concave $h \in \Phi_1$.
Law-invariant convex monetary	Penalized supremum over increasing concave $h \in \Phi_1$.
Law-invariant convex risk functional	Penalized supremum over concave $h \in \Phi_c$, without requiring h increasing or $c = 1$.

That final row is not in the Atlas because the Atlas is monetary. But it is exactly the natural mathematical continuation of the Atlas once monotonicity and unit cash invariance are relaxed.

What The Perverse Indicator Example Shows

Liu, Wang, and Wei (2020) give an example where the inf-convolution of law-invariant preferences is not law invariant. Their counterexample is a functional of the form

$$\rho_F(X) := 1 - \mathbf{1}_{\{X \sim F\}}.$$

It is law invariant because it depends only on whether the distribution of X is exactly F. But it is not a convex risk functional in the Liu sense, and it is not a convex risk measure in the Atlas sense.

First, it fails translation behavior. If $X \sim F$, then $\rho_F(X) = 0$. Usually $X + m \nsim F$ for $m \neq 0$, so $\rho_F(X + m) = 1$. No constant c can make

$$\rho_F(X + m) = \rho_F(X) + cm$$

hold for all m.

Second, it fails convexity. Choose $X \sim F$ and $Y \sim F$ with a dependence structure such that $\lambda X + (1 - \lambda) Y$ does not have distribution F. Then

$$\rho_F(X) = \rho_F(Y) = 0, \qquad \rho_F(\lambda X + (1 - \lambda) Y) = 1,$$

so convexity fails.

Third, it fails continuity. If $X \sim F$ and $X_n = X + 1/n$, then $X_n \to X$ in every L^p norm for bounded X, but generally $X_n \nsim F$. Hence

$$\rho_F(X_n) = 1 \qquad \text{while} \qquad \rho_F(X) = 0.$$

The discontinuity is exactly the kind of pathology that continuity assumptions rule out.

The example therefore lives outside even the extended Liu frameworks. Its value is diagnostic: it shows why continuity, monotonicity, or convexity cannot be treated as cosmetic assumptions when we ask structural questions about inf-convolution. Without them, law invariance alone can encode brittle distributional membership tests rather than stable economic evaluation.

Conclusion

The Atlas can be sharpened by moving the Frittelli and Liu points closer to the front. Frittelli and Rosazza Gianin clarify the connection

$$\text{convexity} + \text{lower semicontinuity} \quad \Longleftrightarrow \quad \text{penalized support representation.}$$

All the familiar risk-measure axioms then appear as restrictions on the support functionals: positivity of weights, unit mass, zero or nonzero penalty, and countable additivity under Fatou-type continuity.

Liu, Cai, Lemieux, and Wang supply the law-invariant continuation:

$$\text{statewise dual density } Z \quad \longmapsto \quad \text{concave rank primitive } h_Z.$$

The Hardy–Littlewood/Frechet–Hoeffding rearrangement step replaces X by its quantile function and replaces Z by its ranked version. Classical distortions, spectral risk measures, Kusuoka mixtures, standard deviation, variance, mean-variance, Gini functionals, and many other examples all sit in the same representation.

The monetary Atlas remains the appropriate for premium principles and acceptability prices. The Liu framework explains the terrain just outside its border. Once the distortion is allowed to be signed, and once $h(1)$ is allowed to differ from one, the same map also covers deviation and variability functionals. That broader view is mathematically cleaner, and it makes the boundary of the monetary theory much easier to see.

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