Continuity of Functionals and Banach vs Topological Vector Spaces

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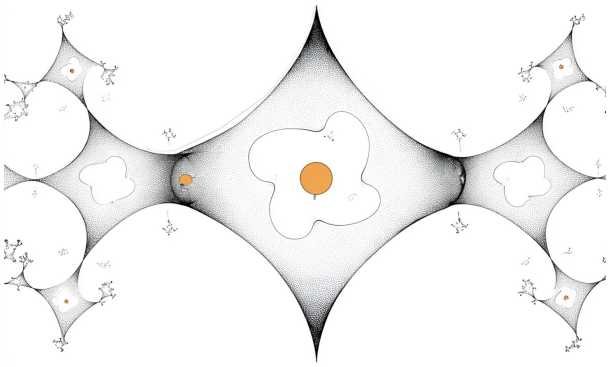

This post has two parts. The first revisits continuity of functionals: above, below, Lebesgue, and Fatou. The second compares topological vector spaces and Banach spaces.

Continuity of Functionals: Limit Theorems for Monotone Functionals

Let ϕ be a monotone functional on a space of random variables, for example L^∞ or L^1. Write $X_n \uparrow X$ when $X_n(\omega)$ increases pointwise a.s. to $X(\omega)$, and similarly $X_n \downarrow X$.

1 Definitions

ϕ is **continuous from below** if

$$X_n \uparrow X \qquad \Longrightarrow \qquad \phi(X_n) \uparrow \phi(X).$$

This is the monotone convergence property for increasing sequences.

ϕ is **continuous from above** if

$$X_n \downarrow X \qquad \Longrightarrow \qquad \phi(X_n) \downarrow \phi(X).$$

This is the decreasing-sequence analogue.

ϕ has the **Lebesgue property** if whenever $X_n \to X$ a.s. and the sequence is order bounded, for example

$$|X_n| \le Y$$

for some fixed Y in the space, then

$$\phi(X_n) \to \phi(X).$$

In many lattice settings this is equivalent to having both continuity from below and continuity from above.

ϕ has the **Fatou property** if whenever $X_n \to X$ a.s. and $|X_n| \le Y$ for some fixed Y, then

$$\phi(X) \le \liminf_{n\to\infty} \phi(X_n).$$

Fatou is lower semicontinuity with respect to dominated a.s. convergence.

2 Implication Structure

For monotone functionals:

$$\text{Lebesgue} \quad \Longrightarrow \quad \begin{cases} \text{continuous from above,} \\ \text{continuous from below,} \\ \text{Fatou.} \end{cases}$$

The first two implications are immediate. For Lebesgue $\Rightarrow$ Fatou, let

$$L_n := \inf_{k \geq n} X_k, \qquad U_n := \sup_{k \geq n} X_k.$$

If $X_n \to X$ a.s., then

$$L_n \uparrow X, \qquad U_n \downarrow X.$$

Monotonicity gives

$$L_n \leq X_k \leq U_n \qquad (k \geq n),$$

hence

$$\phi(L_n) \leq \inf_{k \geq n} \phi(X_k) \leq \sup_{k \geq n} \phi(X_k) \leq \phi(U_n).$$

If ϕ is Lebesgue, then

$$\phi(L_n) \uparrow \phi(X), \qquad \phi(U_n) \downarrow \phi(X),$$

so

$$\phi(X) \leq \liminf \phi(X_n).$$

Lebesgue is strongest condition. In general, none of the remaining three implies the others:

- Continuity from above does not imply continuity from below;
- Continuity from below does not imply continuity from above;
- Continuity from above does not imply Fatou;
- Continuity from below does not imply Fatou;
- Fatou does not imply continuity from above;
- Fatou does not imply continuity from below.

The basic examples are:

- Expectation is fully continuous;
- Essential supremum behaves well for increasing approximation and for Fatou, but badly for decreasing approximation;
- Essential infimum is the mirror image for monotone continuity, and it fails Fatou.
- Upper quartiles are *not* Fatou; lower quartiles are Fatou

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3 Distinguishing Examples

Work on an atomless probability space, say $([0,1], \mathcal{B}, \lambda)$.

3.1 Expectation

Let

$$\phi(X) := \mathsf{P}X = \int X \, dP.$$

Then ϕ has the Lebesgue property, hence all the others.

This is exactly the usual dominated convergence theorem, together with monotone convergence from above and below when integrability is available. Expectation is the model case where everything works.

3.2 Essential supremum

Let

$$\phi(X) := \operatorname{ess\,sup} X.$$

Then:

- ϕ is continuous from below;
- ϕ has the Fatou property;
- ϕ is not continuous from above.

Why continuity from below holds: if $X_n \uparrow X$, then the essential suprema increase to the essential supremum of X.

Why Fatou holds: if $X_n \to X$ a.s., then pointwise

$$X \leq \liminf_n \sup_{k \geq n} X_k,$$

and passing to essential suprema gives

$$\operatorname{ess\,sup} X \leq \liminf_n \operatorname{ess\,sup} X_n.$$

Why continuity from above fails: take

$$X_n := 1_{(0,1/n)}.$$

Then $X_n \downarrow 0$ a.s., but

$$\phi(X_n) = 1 \quad \text{for all } n, \qquad \phi(0) = 0.$$

So

$$\phi(X_n) \not\downarrow \phi(0).$$

This shows:

- Fatou does not imply continuity from above;
- continuity from below does not imply continuity from above;
- Fatou does not imply Lebesgue.

3.3 Essential infimum

Let

$$\phi(X) := \operatorname{ess\,inf} X.$$

Then:

- ϕ is continuous from above;
- ϕ is not continuous from below;
- ϕ does not have the Fatou property.

Why continuity from above holds: if $X_n \downarrow X$, then the essential infima decrease to the essential infimum of X.

Why continuity from below fails: take

$$X_n := -1_{(0,1/n)}.$$

Then $X_n \uparrow 0$ a.s., but

$$\phi(X_n) = -1 \quad \text{for all } n, \qquad \phi(0) = 0.$$

Why Fatou fails: the same example gives

$$\phi(0) = 0 \not\leq \liminf_n \phi(X_n) = -1.$$

This shows:

- continuity from above does not imply continuity from below;
- continuity from above does not imply Fatou;
- continuity from above does not imply Lebesgue.

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4 Summary

functional ϕ	from below	from above	Fatou	Lebesgue
$\mathsf{P}X$	yes	yes	yes	yes
$\operatorname{ess\,sup} X$	yes	no	yes	no
$\operatorname{ess\,inf} X$	no	yes	no	no
Lower quartile	yes	no	yes	no
Upper quartile	no	yes	no	no

Recall

- Lower quartile is left continuous and upper is right continuous.
- Lower quantiles are lower semicontinuous, so they have Fatou and continuity from below;
- Upper quantiles are upper semicontinuous, so they have continuity from above, but not Fatou;
- Lebesgue fails for both because atoms can make the left and right quantiles differ.

The Hierarchy of Spaces: TVS vs. Banach Spaces

In our undergraduate years, we often treat "Vector Space" as a synonym for something with a norm. But as we move into L^p theory and distribution theory, the distinction between a **Topological Vector Space (TVS)** and a **Banach Space** becomes the difference between a functional analyst's playground and a topological minefield.

5 Levels of Niceness

A **Banach Space** is the luxury suite of functional analysis. It requires three distinct layers of structure:

1. **Algebraic:** It is a vector space.
2. **Metric:** It has a norm (a way to measure length).
3. **Analytic:** It is complete (Cauchy sequences converge).

A **Topological Vector Space (TVS)**, by contrast, is the big tent. It only requires that vector addition and scalar multiplication are continuous with respect to the topology. There is no requirement for a norm, a metric, or even convexity.

Feature	Topological Vector Space (General)	Banach Space
Geometry	Defined by open sets.	Defined by a single norm.
Convexity	Not guaranteed.	Always locally convex (balls are convex).
Metrizability	Optional (e.g., the Weak Topology).	Always a complete metric space.
Dual Space	Can be trivial (zero).	Always large (Hahn-Banach).

6 The Nightmare Case: L^p for $0 < p < 1$

For an actuary or a probabilist, L^1 and L^2 are home. But consider $L^p[0,1]$ for $p < 1$. This is a TVS that is not a Banach space (and not even locally convex).

The unit ball $\{f : \int |f|^p < 1\}$ is **concave**. This leads to a catastrophic failure of intuition:

- There are **no** non-zero continuous linear functionals ($X' = \{0\}$).
- The Hahn-Banach theorem fails completely; you cannot separate a point from a convex set with a hyperplane.
- In this world, the Weak Topology doesn't exist because there are no test functions to define it.

7 Fréchet Spaces: Banach-Lite

Sometimes we have a metric but no norm. Consider $C(\mathbb{R})$, the space of continuous functions on the whole real line. We use the topology of uniform convergence on compact sets.

This is a **Fréchet Space**: it is complete and metrizable, but because the importance of the function's behavior depends on the domain size, no single norm can capture the topology. It is stronger than a general TVS but lacks the rigid unit-ball geometry of a Banach space.

8 The Issue with Bases

Every vector space has a **Hamel (algebraic) basis**, but in a Banach space, this basis is usually an uncountable mess. We prefer a **Schauder basis**—a countable sequence that allows for unique series representations $x = \sum \alpha_n e_n$.

However, being a Banach space doesn't guarantee a Schauder basis.

1. Separability: Only separable spaces can even dream of having a Schauder basis (this excludes L^∞).
2. The Enflo Counter-example: In 1973, Per Enflo famously proved that there exist separable Banach spaces with no Schauder basis.

Without a basis, the space lacks the Approximation Property. You can't necessarily approximate every compact operator with a finite-rank one. In these spaces, the idea that an operator is just an infinite matrix completely breaks down.

9 Around the Edges

If you are working in L^p ($1 \leq p < \infty$) or a Hilbert space, you are in a Banach space paradise where every point can be separated, every ball is convex, and (usually) a basis exists. Once you move to L^p ($p < 1$) or distribution spaces (D'), the norm vanishes, the unit ball collapses, and you are left with the raw, structural machinery of the Topological Vector Space.