Convergence and Counter-Examples for Risk Measures

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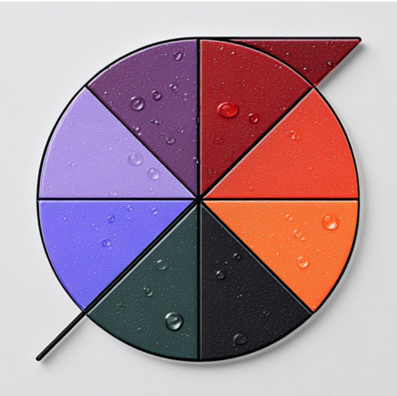

Background on liminf and limsup

The definitions are

$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty}\left(\inf_{k\ge n} a_k\right)$$

and

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty}\left(\sup_{k\ge n} a_k\right).$$

We always have $\liminf a_n \le \limsup a_n$.

If the ordinary limit $\lim_{n\to\infty} a_n$ exists (as a real number), then

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n.$$

So you cannot have $\liminf a_n < \lim a_n$ when $\lim a_n$ exists. The point of $\liminf$ and $\limsup$ is that they are always defined, even when $\lim a_n$ does not exist.

For example, if $a_n = (-1)^n$, then the sequence oscillates between -1 and 1. For any tail $\{a_k : k \ge n\}$ we have $\inf_{k\ge n} a_k = -1$ and $\sup_{k\ge n} a_k = 1$, so

$$\liminf_{n\to\infty} a_n = -1, \qquad \limsup_{n\to\infty} a_n = 1,$$

and $\lim_{n\to\infty} a_n$ does not exist. In Fatou-type statements we use $\liminf$ because the sequence $\pi(X_n)$ need not converge; $\liminf$ still captures the eventual lower level of the sequence and is the correct notion for "no upward jump" in the limit.

Assumptions

We use the actuarial sign convention, X is a loss (higher is worse), and a pricing / risk measure $\pi(X)$ is a required premium / capital (higher is worse).

Sets are identified with their indicator functions, so A is a set and stand for the function

$$A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

often written as 1_A.

Definitions

A monotone pricing / risk measure $\pi : L^\infty \to \mathsf{R}$ satisfies the **Fatou property** if, whenever $X_n \to X$ almost surely and $\sup_n \|X_n\|_\infty < \infty$, we have

$$\pi(X) \le \liminf_{n\to\infty} \pi(X_n).$$

The interpretation is that if X_n approximates X in a strong pointwise sense (a.s.), then the price / required capital for X should not be a nasty upward surprise relative to the eventual behavior of the prices of the approximations. There is no "jump up" in risk.

The bounded assumption is needed: consider $X_n = -n[0, 1/n]$ a sequence of functions with expectation -1 that converges pointwise to 0.

Two related regularity properties are continuity from below and continuity from above. **Continuity from below** means that if $X_n \uparrow X$ a.s., then

$$\pi(X_n) \uparrow \pi(X).$$

Continuity from above means that if $X_n \downarrow X$ a.s., then

$$\pi(X_n) \downarrow \pi(X).$$

The **Lebesgue property** (also called **order continuity**) is stronger: if $X_n \to X$ a.s. and there exists $Y \in L^\infty$ with $|X_n| \le Y$ for all n, then

$$\pi(X_n) \to \pi(X).$$

Lebesgue implies Fatou, and for monotone π it implies CFB and CFA.

Heuristically, Fatou rules out upward jumps under bounded a.s. limits, continuity from below rules out upward jumps along increasing approximations, continuity from above rules out downward jumps along decreasing approximations, and the Lebesgue property gives full convergence under dominated a.s. convergence.

For $p \in (0, 1)$, define the **lower** and **upper** ***p*-quantiles** by

$$q_p^-(X) := \inf\{x \in \mathsf{R} : \mathsf{P}(X \le x) \ge p\}, q_p^+(X) := \inf\{x \in \mathsf{R} : \mathsf{P}(X \le x) > p\}.$$

When used as pricing / risk measures under the actuarial sign convention, $\pi(X) = q_p^-(X)$ is a "lower-quantile premium," and $\pi(X) = q_p^+(X)$ is the corresponding strict / upper-quantile premium.

Examples

We show:

1. For monotone, Fatou is equivalent to continuous from below.
2. Lower quantile has Fatou.
3. Upper quantile does not have Fatou and is not CFB.
4. Cofinite / Poisson is not continuous from below and therefore fails Fatou.
5. Cofinite (Fréchet filter) is not continuous from above.
6. Ess sup is not Lebesgue, not CFA, but it is Fatou.
7. Ess sup does not achieve sup in dual representation.

Three well known items that we do not show.

a. TVaR has Fatou, continuity from below, and the Lebesgue property. It is the "good" behavior benchmark.
b. Jouini et al. (2006) / Svindland (2010): a **law invariant** monetary convex risk measures on an atomless probability space **has the Fatou property**.
c. Generally, a convex monetary risk measure is Fatou iff there is a robust dual representation over σ-additive measures (rather than finitely additive charges).

For monotone, Fatou iff continuous from below. Let $\pi : L^\infty \to \mathsf{R}$ be monotone, and assume π has the Fatou property. Suppose $X_n \uparrow X$ a.s. and $\sup_n \|X_n\|_\infty < \infty$.

By monotonicity, $\pi(X_n) \le \pi(X)$ for all n, so the nondecreasing sequence $\pi(X_n)$ has a limit

$$L := \lim_{n\to\infty} \pi(X_n) \le \pi(X).$$

Since $X_n \to X$ a.s. and the sequence is uniformly bounded, the Fatou property gives

$$\pi(X) \le \liminf_{n\to\infty} \pi(X_n).$$

But $\pi(X_n)$ is nondecreasing, so $\liminf_{n\to\infty} \pi(X_n) = \lim_{n\to\infty} \pi(X_n) = L$. Hence

$$\pi(X) \le L \le \pi(X),$$

so $L = \pi(X)$, i.e.

$$\pi(X_n) \uparrow \pi(X).$$

Therefore π is continuous from below.

Conversely, if $X_n \to X$ a.s. is bounded, define the tail inf sequence $Y_n = \inf_{k\ge n} X_k$. Since $X_n \to X$, $Y_n \uparrow X$ and so by CFB $\pi(Y_n) \uparrow \pi(X)$. By construction $Y_n \le X_k$ for all $k \ge n$ and so $\pi(Y_n) \le \pi(X_k)$, so $\pi(Y_n) \le \inf_{k\ge n} \pi(X_k)$. Taking limits gives

$$\pi(X) = \lim \pi(Y_n) \le \lim(\inf_{k\ge n} \pi(X_k)) = \liminf \pi(X_n)$$

giving Fatou.

The **lower quantile has the Fatou property**. Suppose $X_n \to X$ a.s. and $\sup_n \|X_n\|_\infty < \infty$. Then X_n converges in distribution to X, and by Portmanteau, for each closed set F,

$$\mathsf{P}(X \in F) \ge \limsup_{n\to\infty} \mathsf{P}(X_n \in F).$$

Taking $F = (-\infty, x]$ gives

$$\mathsf{P}(X \le x) \ge \limsup_{n\to\infty} \mathsf{P}(X_n \le x).$$

Fix any $x < q_p^-(X)$. By definition of $q_p^-(X)$, we have $\mathsf{P}(X \le x) < p$, hence

$$\limsup_{n\to\infty} \mathsf{P}(X_n \le x) \le \mathsf{P}(X \le x) < p,$$

so for all sufficiently large n, $\mathsf{P}(X_n \le x) < p$. Therefore $x \notin \{y : \mathsf{P}(X_n \le y) \ge p\}$ eventually, so $q_p^-(X_n) > x$ eventually, which implies

$$\liminf_{n\to\infty} q_p^-(X_n) \ge x.$$

Since this holds for every $x < q_p^-(X)$, we conclude

$$q_p^-(X) \le \liminf_{n\to\infty} q_p^-(X_n),$$

which is the Fatou property for $\pi(X) = q_p^-(X)$.

The **upper quantile does not have the Fatou property**. Work on $([0,1], \mathcal{B}, \lambda)$, take $p = \frac{1}{2}$, and define

$$X(\omega) = \mathbf{1}_{(1/2,1]}(\omega).$$

Choose disjoint measurable sets $B_n \subset (1/2, 1]$ with

$$\mathsf{P}(B_n) = 2^{-(n+1)},$$

which is feasible because $\sum_{n\ge1} 2^{-(n+1)} = \frac{1}{2}$ and Lebesgue measure is non-atomic. Define the indicator variables

$$X_n(\omega) = ((1/2, 1] \setminus B_n)(\omega).$$

Then $0 \le X_n \le 1$ and, since the B_n are disjoint, each ω lies in at most one B_n, so $X_n(\omega) \to X(\omega)$ a.s.

Now compute the upper quantile. For each n,

$$\mathsf{P}(X_n \le 0) = \mathsf{P}([0, 1/2] \cup B_n) = \frac{1}{2} + 2^{-(n+1)} > \frac{1}{2},$$

so $q_{1/2}^+(X_n) = 0$. For the limit,

$$\mathsf{P}(X \le 0) = \frac{1}{2} \not> \frac{1}{2},$$

so $x = 0$ does not qualify in the definition of $q_{1/2}^+$, and the smallest x with $\mathsf{P}(X \le x) > \frac{1}{2}$ is $x = 1$, hence $q_{1/2}^+(X) = 1$. Therefore

$$q_{1/2}^+(X) = 1 \quad \text{but} \quad \liminf_{n\to\infty} q_{1/2}^+(X_n) = 0,$$

so **Fatou fails** for $\pi(X) = q_{1/2}^+(X)$.

The previous example shows Fatou failing for a non-monontonic sequence. We can create a simpler, monotonic failure of CFB and Fatou as follows. Take

$$X_n(\omega) = \begin{cases} 0 & \omega \in [0, 0.5 + 1/(n+2)] \\ 1 & \omega \in (0.5 + 1/(n+2), 1] \end{cases}$$

then $X_n \uparrow X$, $\mathsf{P}(X_n \le 0) = 0.5 + 1/(n+2) > 0.5$ and so $q^+_{1/2}(X_n) = 0$. But $q^+_{1/2}(X) = 1$. Hence **CFB fails**.

A different source of **Fatou failure** comes from finitely additive "probabilities." Let $\Omega = \{0, 1, 2, \dots\}$. Define a finitely additive set function Q on 2^Ω by

$$Q(S) = \begin{cases} 0, & S \text{ finite,} \\ 1, & S \text{ cofinite.} \end{cases}$$

and extend Q (in any way) to all sets so that it remains finitely additive and satisfies $Q(\Omega) = 1$. Define a functional by $\pi(X) = Q(X)$ for bounded X (in particular for indicators). Let

$$X_n = \{1, \dots, n\}.$$

Then $X_n \uparrow \Omega$ pointwise and $0 \le X_n \le 1$. Each X_n has finite support, so $\pi(X_n) = Q(\{1, \dots, n\}) = 0$, but $\pi(\Omega) = Q(\Omega) = 1$. Thus

$$\pi\Big(\lim_{n\to\infty} X_n\Big) = 1 > \lim_{n\to\infty} \pi(X_n) = 0,$$

so **continuity from below fails**, and therefore **Fatou fails** as well. This example is not law invariant wrt any P that assigns mass to atome. E.g., if P is geometric 1/2, then $A = \{1\}$ has $\pi(A) = 0$ because it is finite. $B = A^c$ has $\pi(B) = 1$ because it is infinite. But A and B have the same law: under P they both have probability 1/2.

The same functional is **not continuous from above**. Define the tail sets

$$S_n = \{n, n+1, n+2, \dots\}.$$

Then $S_{n+1} \subset S_n$, so the indicator $S_n \downarrow 0$ pointwise, hence $S_n \downarrow 0$ a.s. Moreover, each S_n is cofinite since $\Omega \setminus S_n = \{1, \dots, n-1\}$ is finite, so

$$\pi(X_n) = Q(S_n) = 1 \qquad \text{for all } n,$$

while

$$\pi(0) = \pi(\emptyset) = Q(\emptyset) = 0.$$

Thus π is **not continuous from above**: even though $S_n \downarrow 0$, we do not have $\pi(S_n) \downarrow \pi(0)$.

Now define the worst-case pricing functional

$$\pi(X) := \operatorname{ess\,sup} X$$

under a non-atomic P (for example Lebesgue measure on $[0,1]$). This functional admits a dual representation of the form

$$\operatorname{ess\,sup} X = \sup\{\mathsf{Q}(X) : \mathsf{Q} \ll \mathsf{P}\},$$

where the supremum ranges over countably additive probabilities Q absolutely continuous with respect to P. The supremum need not be achieved: one can choose Q with densities increasingly concentrated on sets where X is near its essential supremum, but there is typically no absolutely continuous Q that concentrates all mass on a point or on an arbitrarily small set. This is a sup-versus-max issue (attainment), and it is logically separate from Fatou and other convergence results.

Let $\pi(X) = \operatorname{ess\,sup}(X)$. Then π **has Fatou**. Suppose $X_n \to X$ almost surely (and boundedly). Let $L = \liminf_{n\to\infty} \pi(X_n)$. We want to show $\pi(X) \leq L$. Pass to a subsequence n_k such that $\lim_{k\to\infty} \pi(X_{n_k}) = L$ and let $c_k = \pi(X_{n_k}) = \operatorname{ess\,sup}(X_{n_k})$.

By definition of the essential supremum, for each k, we have

$$X_{n_k} \leq c_k \quad \text{a.s.}$$

Since the inequality holds almost surely for each k, it holds almost surely for the countable intersection of the sets where it holds. Taking the limit as $k \to \infty$:

$$X = \lim_{k\to\infty} X_{n_k} \leq \lim_{k\to\infty} c_k = L \quad \text{a.s.}$$

Since $X \leq L$ almost surely, by definition of the essential supremum: $\operatorname{ess\,sup}(X) \leq L$.

Finally, ess sup **does not have the Lebesgue property**. With P Lebesgue on $[0, 1]$, let

$$X_n = [0, 1/n].$$

Then $X_n \downarrow 0$ a.s. and $0 \leq X_n \leq 1$, but

$$\pi(X_n) = \operatorname{ess\,sup} X_n = 1 \quad \text{for all } n,$$

but $\pi(0) = 0$. So π is **not CFA**. Hence $\pi(X_n) \not\to \pi(0)$, so dominated a.s. convergence does not imply convergence of prices. This is a failure of the Lebesgue property (and, in particular, failure of continuity from above), even though ess sup does satisfy the Fatou property.

References

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