Spectral Risk Measures are not Weak Acceptance Time Consistent

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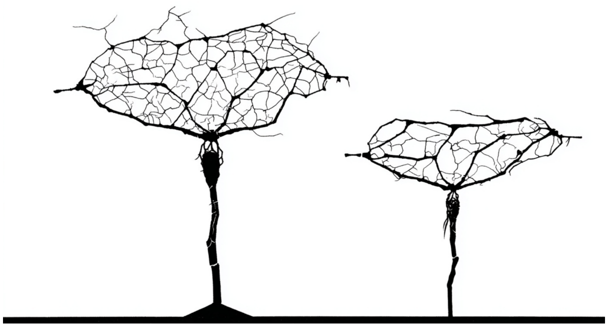

This note provides a short and motivated proof of Theorem 4.6 from Bielecki et al. (2024), showing that the risk measure associated with a strictly concave distortion on an atomless probability space is not weak acceptance time consistent.

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be an atomless probability space. Then for every $p \in (0,1)$ there exists $B \in \mathcal{F}$ with $\mathsf{P}(B) = p$, hence $(\Omega, \mathcal{F}, \mathsf{P})$ supports Bernoulli(p) variables (Föllmer and Schied 2016, Appendix A27). Let $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 = \mathcal{F}$ be a filtration at times $t \in \{0,1,2\}$. Let $g : [0,1] \to [0,1]$ be a concave distortion with $g(0) = 0$ and $g(1) = 1$.

For a bounded profit and loss variable X, the dynamic distortion risk measure of Bielecki et al. (2024) is the conditional Choquet integral

$$\rho_t^g(X) = \int_0^\infty g(\mathsf{P}(-X > y \mid \mathcal{F}_t))\, dy + \int_{-\infty}^0 (g(\mathsf{P}(-X > y \mid \mathcal{F}_t)) - 1)\, dy.$$

Weak acceptance time consistency (WATC) for ρ means: for $s > t$,

$$\rho_s^g(X) \le 0 \;\Rightarrow\; \rho_t^g(X) \le 0.$$

For a nonnegative bounded loss $L \ge 0$, define the induced price functional

$$\pi_t^g(L) := \rho_t^g(-L) = \int_0^\infty g(\mathsf{P}(L > y \mid \mathcal{F}_t))\, dy,$$

since the second integral vanishes for $X = -L$.

By cash additivity, $\rho_t^g(m - L) = \pi_t^g(L) - m$. Hence WATC is equivalent to: for all bounded $L \ge 0$ and all constants m, if $\pi_s^g(L) \le m$ a.s., then $\pi_t^g(L) \le m$ a.s.

Intuitively, WATC can fail even in a two-step tree. Consider a mixture of two conditional loss distributions, one low-spread and one high-spread, with the mixture component revealed at $t = 1$. Choose them so that the time-1 conditional distortion price is the same on both branches, so $\rho_1(m - L) = 0$ statewise. At $t = 0$, however, ρ_0 applies the distortion to the unconditional mixture distribution, which exposes more extreme tail behavior than either conditional law alone. The theorem below formalizes this mechanism and shows that any non-trivial concave distortion yields such a counterexample, and hence fails WATC.

Theorem 0.1 (Nontrivial concave distortions fail WATC (Bielecki et al. 2024 Theorem 4.6))**.** *Let g be a concave distortion with $g(0) = 0$, $g(1) = 1$. Assume g is neither the identity ($g(s) = s$) not the max distortion ($g_{\max}(s) = \{s > 0\}$ indicator function). Then there exists a two-step filtered two-branch tree embedded in $(\Omega, \mathscr{F}, \mathsf{P})$, and a bounded X such that*

$$\rho_1^g(X) \le 0 \text{ a.s.} \quad \text{but} \quad \rho_0^g(X) > 0,$$

so ρ^g is not weakly acceptance time consistent.

Proof. By assumption, there exist $r, s \in (0, 1)$ such that $g(sr) > sg(r)$. (Otherwise $g(sr) = s\,g(r)$ for all r, s, forcing $g(u) = u$. Here we also rely on $g \ne g_{\max}$ for strict inequality.)

Fix such (r, s) and set $w := g(r) \in (0, 1)$.

Next, we build a tree and match time-1 prices. Let S be a Bernoulli random variable with $\mathsf{P}\{S = 1\} = s$, and let $\mathscr{F}_1 = \sigma(S)$. On each state, let the loss L take two values with the same conditional probabilities:

$$\begin{cases} \mathsf{P}\{L = b_1 \mid S = 0\} & = r \\ \mathsf{P}\{L = a_1 \mid S = 0\} & = 1 - r \end{cases} \qquad \text{and} \qquad \begin{cases} \mathsf{P}\{L = b_2 \mid S = 1\} & = r \\ \mathsf{P}\{L = a_2 \mid S = 1\} & = 1 - r \end{cases}$$

with $a_2 < a_1 < b_1 < b_2$.

For a two-point loss $Y \in \{a, b\}$ with $\mathsf{P}\{Y = b\} = r$, one has

$$\pi^g(Y) = a + (b - a)w.$$

Hence, for any $m > 0$ and any choice of $a \in (0, m)$, setting

$$b = \frac{m - (1 - w)a}{w}$$

gives $\pi^g(Y) = m$. Choose $0 < a_2 < a_1 < m$ and define b_1, b_2 by this formula; then $a_2 < a_1 < b_1 < b_2$ and

$$\pi_1^g(L) = m$$

in each state $S = 0, 1$.

We now show the time-0 price strictly exceeds m. Unconditionally, L has four atoms with probabilities

$$s(1 - r),\ (1 - s)(1 - r),\ (1 - s)r,\ \ sr$$

on the ordered values $a_2 < a_1 < b_1 < b_2$. Writing out $\pi_0^g(L)$ as the integral of survival probabilities on the three intervals (a_2, a_1), (a_1, b_1), (b_1, b_2) yields

$$\pi_0^g(L) = m + (a_1 - a_2)D,$$

where

$$D := g(1 - s(1 - r)) - 1 + \frac{1 - w}{w} g(sr).$$

Concavity gives the chord bound $g(1 - s(1 - r)) \geq 1 - s + sw$ (mixing 1 and r), and the choice of (r, s) gives $g(sr) > sw$. Substituting,

$$D > (1 - s + sw) - 1 + \frac{1 - w}{w} sw = 0,$$

so $\pi_0^g(L) > m$.

Finally, let $X := m - L$. Then

$$\begin{aligned} \rho_1^g(X) &= \pi_1^g(L) - m = 0 \leq 0 \\ \rho_0^g(X) &= \pi_0^g(L) - m > 0, \end{aligned}$$

showing that WATC fails. □

Remark 0.1.

1. Bielecki et al. (2024) parameterize distortions via a mixing measure μ (Kusuoka/spectral representation). This is an equivalent encoding of the same spectral functional.
2. The counterexample requires $g(r) \in (0, 1)$ and a strict chord inequality at some tail level. It fails only for the two degenerate endpoint cases: the identity and the max distortion $g_{\max}$ (which yields essential supremum for nonnegative losses).
3. The argument uses only concavity, so it covers discontinuous distortions (e.g. with $g(0+) > 0$) and does not require absolute continuity.

References.

Bielecki, T. R., Cialenco, I., & Liu, H. (2024). Time consistency of dynamic risk measures and dynamic performance measures generated by distortion functions. *Stochastic Models*. https://doi.org/10.1080/15326349.2024.2353045

Föllmer, H., & Schied, A. (2016). *Stochastic Finance: An Introduction in Discrete Time* (Fourth.). Berlin, Boston: Walter de Gruyter. https://doi.org/10.1017/CBO9781107415324.004